Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of Geometry and Physics 51 (2004) 332–352

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

Conservation laws for some nonlinear evolution equations which describe pseudo-spherical surfaces

A.H. Khater [a,b,∗], D.K. Callebaut [b], S.M. Sayed [a]

[a] *Mathematics Department, Faculty of Science, Cairo University, Beni-Suef, Egypt*
[b] *Dept. Nat., Campus Drie Eiken, University of Antwerp, B-2610 Antwerp, Belgium*

Received 20 January 2003; received in revised form 27 October 2003

Available online 22 January 2004

Abstract

The relation between pseudo-spherical surfaces and the inverse scattering method is exemplified for several evolution equations. Conservation laws for the latter ones are obtained using a geometrical property of these pseudo-spherical surfaces.

MSC: 37K10; 58A10; 65M32; 14Q10; 35Q51

JGP SC: Differential forms; Solitons; Inverse problems; Surfaces; Hyper-surfaces; Differential geometry

Keywords: Conservation laws; Pseudo-spherical surfaces; Nonlinear evolution equations

1. Introduction

In 1979, Sasaki [1] observed that a class of nonlinear partial differential equations (NLPDEs), such as Korteweg–de Vries (KdV), modified Korteweg–de Vries (MKdV) and sine-Gordon (SG) equations which can be solved by the Ablowitz, Kaup, Newell and Segur (AKNS) 2×2 inverse scattering method (ISM) [2], was related to pseudo-spherical surfaces (pss). The geometric notion of a differential equation (DE), for a real function, which describes a pss was actually introduced in the literature by Chern and Tenenblat [3], where equations of type $u_t = F(u, u_x, \ldots, u_xk)$ $(u_xk = \partial^k u/\partial x^k)$ were studied systematically. Later, in [4,5], this concept was applied to other types of DEs. A generic solution of such an equation provides a metric defined on an open subset in R^2, for which the Gaussian curvature is -1.

∗ Corresponding author.
E-mail address: khater_ah@yahoo.com (A.H. Khater).

0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2003.11.009

Such a DE is characterized as being the integrability condition of a linear problem of the form

$$v_x = \left(\begin{pmatrix} \eta/2 & 0 \\ 0 & -\eta/2 \end{pmatrix} + A\right) v, \qquad v_t = Qv,$$

where η is a parameter, Q is a 2×2 traceless matrix and A is a 2×2 off-diagonal matrix depending on η, u and its derivatives. Examples of this class of equations are (real) equations of the AKNS type. Other examples, which are not AKNS, can be found in [4–6]. Geometric interpretation of special properties (such as infinite number of conservation laws and Bäcklund transformations (BTs)) for solutions of DEs which describe pss have been systematically exploited in [7–10]. In 1995, Kamran and Tenenblat [11], extending the results of Chern and Tenenblat [3], gave a complete classification of the evolution equations (EEs) of type $u_t = F(u, u_x, \ldots, u_x^k)$ which describe pss by considering equations which are the integrability condition of a linear problem of the form given above.

Moreover, they proved that there exists, under a technical assumption, a smooth mapping transforming any generic solution of one such equation into a solution of the other. This geometric notion of scalar DEs was also generalized to DEs of the type $F(x, t, u, u_x, \ldots, u_{x^n t^m}) = 0$ by Reyes recently in [8,21,22].

Let g be a Riemannian metric on M^2, ∇ the corresponding Levi–Civita connection on the tangent bundle TM^2, $\{e_1, e_2\}$ be a moving orthonormal frame on some open domain $U \subset M^2$ and $\{\omega_1, \omega_2\}$ a corresponding moving coframe. The relations $\nabla(e_i) = \omega_i^j \otimes e_j$ define the connection one-form matrix ω_i^j with respect to the frame $\{e_1, e_2\}$. The orthogonality of this frame implies that $\omega_1^1 = \omega_2^2 = 0, \omega_1^2 = -\omega_2^1 = (\omega_3)$. Hence the Levi–Civita connection one-form on the tangent bundle TM^2 with respect to the moving frame $\{e_1, e_2\}$ is

$$\begin{pmatrix} 0 & \omega_3 \\ -\omega_3 & 0 \end{pmatrix}.$$

It yields the following structural equations:

$$d\omega_1 = \omega_3 \wedge \omega_2, \quad d\omega_2 = \omega_1 \wedge \omega_3. \tag{1}$$

The Gaussian curvature k of the space M^2 is defined by the Gauss equation

$$d\omega_3 = k\omega_2 \wedge \omega_1. \tag{2}$$

Sasaki [1] gave a formula for some local connection on a two-dimensional real Riemannian manifold M^2, which is quite relevant in the theory of nonlinear integrable partial differential equations:

$$\Omega = \frac{1}{2}\begin{pmatrix} \omega_2 & \omega_1 - \omega_3 \\ \omega_1 + \omega_3 & -\omega_2 \end{pmatrix}, \tag{3}$$

as a new connection form for some (nonspecified) bundle over M^2. The key property of the matrix one-form Ω is that it satisfies the curvature condition

$$\Theta \equiv \mathrm{d}\Omega - \Omega \wedge \Omega = 0,$$

iff $k = -1$ on U. In some older (for example [12]) and many subsequent papers (some of the most recent are [13–16]) different matrix one-forms were discussed, depending on a function u (or some functions) of some independent variables, such that the curvature condition $\Theta = 0$ for this form is equivalent to one of the well known NLPDEs having an infinite number of conservation laws and symmetry groups. The generalization to higher dimensions is given in [17].

The condition $\Theta = 0$ depends only on relation (1) and the commutative relations in the algebra $\mathrm{SL}(2, R)$. The one-form Ω may be written as [17]:

$$\Omega = \begin{pmatrix} 0 & \omega_3 & -\omega_1 \\ -\omega_3 & 0 & -\omega_2 \\ -\omega_1 & -\omega_2 & 0 \end{pmatrix},$$

which contains the Levi–Civita connection form

$$\begin{pmatrix} 0 & \omega_3 \\ -\omega_3 & 0 \end{pmatrix},$$

as a direct summand and avoids the surprising factor 1/2 in (3).

As a consequence, each solution of the DE provides a metric on M^2, whose Gaussian curvature is constant, equal to -1. Moreover, the above definition is equivalent to saying that the DE for u is the integrability condition for the problem:

$$\mathrm{d}\nu = \Omega\nu, \quad \nu = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}, \tag{4}$$

where d denotes exterior differentiation, ν is a vector and the 3×3 matrix $\Omega(\Omega_{ij}, i, j = 1\text{–}3)$ is traceless

$$\mathrm{tr}\Omega = 0, \tag{5}$$

and consists of a one-parameter (η, the eigenvalue) family of one-forms in the independent variables (x, t), the dependent variable u and its derivatives. Integrability of Eq. (4) requires:

$$0 = \mathrm{d}^2\nu = \mathrm{d}\Omega\nu - \Omega \wedge \mathrm{d}\nu = (\mathrm{d}\Omega - \Omega \wedge \Omega)\nu,$$

or the vanishing of the two-form

$$\Theta \equiv \mathrm{d}\Omega - \Omega \wedge \Omega = 0, \tag{6}$$

which corresponds, by construction, to the original NLEE to be solved. Eq. (4) corresponds to three equations and only selected solutions are possible, i.e. those satisfying (6). This was of course, equally true in that Sasaki formulation.

The main aim of this paper is to extend somewhat the inverse scattering problem in reference [18] by considering ν as a three component vector and Ω as a traceless 3×3 matrix one-form. An application of the original Chern and Tenenblat construction of conservation laws is given for the nonlinear Schrödinger equation (NLSE) [23–28], Liouville's equation [29], two families of equations [10], the Ibragimov–Shabat (IS) equation, the Camassa–Holm (CH) equation and Hunter–Saxton (HS) equation, which are very useful in several areas of physics as may be seen from the numerous references.

The paper is organized as follows. In Section 2 we introduce the inverse scattering problem and apply the geometrical method to several PDEs which describe pss. In Section 3 we obtain an infinite number of conserved densities for some NLEEs which describe pss using a theorem of Cavalcante and Tenenblat [7]. In Section 4 we obtain conservation laws by extending the classical discussion of Wadati et al. [20]. Section 5 contains the conclusion.

2. Inverse scattering problem and DEs which describe pss

Some NLPDEs, invariant to translations in x and t, can be solved exactly by an ISM with the set of linear equations

$$\nu_{1x} = f_{31}\nu_2 - f_{11}\nu_3, \quad \nu_{2x} = -f_{31}\nu_1 - \eta\nu_3, \quad \nu_{3x} = -f_{11}\nu_1 - \eta\nu_2, \tag{7}$$

$$\nu_{1t} = f_{32}\nu_2 - f_{12}\nu_3, \quad \nu_{2t} = -f_{32}\nu_1 - f_{22}\nu_3, \quad \nu_{3t} = -f_{12}\nu_1 - f_{22}\nu_2. \tag{8}$$

The functions f_{ij}, $1 \le i \le 3, 1 \le j \le 2$, depend on x, t, u and its derivatives. They can also be functions of the parameter η. We have restricted ourselves to the case where $f_{21} = \eta$ is a parameter. The compatibility conditions for Eqs. (7) and (8), obtained by cross differentiation, are:

$$f_{12,x} - f_{11,t} = f_{31}f_{22} - \eta f_{32}, \tag{9}$$

$$f_{22,x} = f_{11}f_{32} - f_{12}f_{31}, \tag{10}$$

$$f_{32,x} - f_{31,t} = f_{11}f_{22} - \eta f_{12}. \tag{11}$$

In order to solve (9)–(11) for f_{ij} in general, one finds that another condition still has to be satisfied. This latter condition is the evolution equation (EE). In terms of exterior differential forms the inverse scattering problem can be formulated as follows: Eqs. (7) and (8) can be rewritten in matrix form by using Eq. (4), and Ω is a traceless 3×3 matrix of one-forms given by

$$\Omega = \begin{pmatrix} 0 & f_{31}\,\mathrm{d}x + f_{32}\,\mathrm{d}t & -f_{11}\,\mathrm{d}x - f_{21}\,\mathrm{d}t \\ -f_{31}\,\mathrm{d}x - f_{32}\,\mathrm{d}t & 0 & -\eta\,\mathrm{d}x - f_{22}\,\mathrm{d}t \\ -f_{11}\,\mathrm{d}x - f_{21}\,\mathrm{d}t & -\eta\,\mathrm{d}x - f_{22}\,\mathrm{d}t & 0 \end{pmatrix}. \tag{12}$$

In this scheme (6), yielding the EE, must be satisfied for the existence of a solution $f_{ij}, 1 \le i \le 3, 1 \le j \le 2$. Whenever the functions are real, Sasaki [1] gave a geometrical interpretation for the problem. Consider the one-forms defined by

$$\omega_1 = f_{11}\,\mathrm{d}x + f_{12}\,\mathrm{d}t, \quad \omega_2 = f_{21}\,\mathrm{d}x + f_{22}\,\mathrm{d}t, \quad \omega_3 = f_{31}\,\mathrm{d}x + f_{32}\,\mathrm{d}t.$$

We say that a DE for $u(x, t)$ describes a pss if it is a necessary and sufficient condition for the existence of functions f_{ij}, $1 \le i \le 3$, $1 \le j \le 2$, depending on u and its derivatives, $f_{21} = \eta$, such that the one-forms in Eq. (2), satisfy the structure Eq. (3) of a pss. It follows from this definition that for each nontrivial solution u of the DE, one gets a metric defined on M^2, whose Gaussian curvature is -1.

It has been known, for a long time, that the SG equation describes a pss. More recently, other equations, such as KdV and MKdV equations, were also shown to describe such surfaces [2]. Here we show that other equations such as the NLSE, Liouville's equation, two families of equations, IS equation, CH equation and HS equation describe pss as well. The latter equations proved to be of great importance in many physical applications [3,7,21–30]. The procedure is clarified in the following examples.

Let M^2 be a differentiable surface, parametrized by coordinates x, t.

2.1. Nonlinear Schrödinger equation

Consider

$$\omega_1 = \tfrac{1}{2}(u - u^*)\,\mathrm{d}x + [i\eta(u - u^*) + i(u_x + u_x^*)]\,\mathrm{d}t, \qquad \omega_2 = \eta\,\mathrm{d}x + [2i\eta^2 + iuu^*]\,\mathrm{d}t,$$
$$\omega_3 = \tfrac{-1}{2}(u + u^*)\,\mathrm{d}x + [-i\eta(u + u^*) + i(u_x^* - u_x)]\,\mathrm{d}t. \tag{13}$$

M^2 is a pss iff u satisfies the NLSE

$$iu_t + 2u_{xx} + u^2u^* = 0. \tag{14}$$

2.2. Liouville's equation

Consider

$$\omega_1 = u_x\,\mathrm{d}x, \qquad \omega_2 = \eta\,\mathrm{d}x + \frac{e^u}{\eta}\,\mathrm{d}t, \qquad \omega_3 = \frac{e^u}{\eta}\,\mathrm{d}t. \tag{15}$$

Then M^2 is a pss iff u satisfies the Liouville's equation

$$u_{xt} = e^u. \tag{16}$$

2.3. The family of equations

Consider

$$\omega_1 = -\frac{\xi}{\eta}g'\,\mathrm{d}t, \qquad \omega_2 = \eta\,\mathrm{d}x + \left(\frac{\xi^2 g - \theta}{\eta} + \beta\eta\right)\mathrm{d}t,$$
$$\omega_3 = \xi u_x\,\mathrm{d}x + \xi(\alpha g + \beta)u_x\,\mathrm{d}t. \tag{17}$$

Then M^2 is a pss iff u satisfies the family of equations

$$[u_t - (\alpha g(u) + \beta)u_x]_x = -g'(u), \tag{18}$$

where $g(u)$ is a differentiable function of u which satisfies $g'' + \mu g = \theta$, where α, β, μ and θ are real constants, such that $\xi^2 = \alpha\eta^2 + \mu$.

2.4. *The family of equations*

Consider

$$\omega_1 = \xi u_x \,\mathrm{d}x + \xi(\alpha g + \beta)u_x \,\mathrm{d}t, \qquad \omega_2 = \eta \,\mathrm{d}x + \left(\frac{\xi^2 g - \theta}{\eta} + \beta\eta\right) \mathrm{d}t,$$
$$\omega_3 = \frac{\xi}{\eta} g' \,\mathrm{d}t. \tag{19}$$

Then M^2 is a pss iff u satisfies the family of equations

$$[u_t - (\alpha g(u) + \beta)u_x]_x = g'(u), \tag{20}$$

where $g(u)$ is a differentiable function of u which satisfies $g'' + \mu g = \theta$, where α, β, μ and θ are real constants, such that $\xi^2 = \alpha\eta^2 - \mu$.

2.5. *NLEE*

Consider

$$\omega_1 = -\eta\sqrt{\tfrac{2}{3}}u_x \,\mathrm{d}t, \qquad \omega_2 = \eta \,\mathrm{d}x + \left(\eta^3 + \tfrac{1}{3}\eta u^2 + \alpha\eta\right) \mathrm{d}t,$$
$$\omega_3 = \sqrt{\tfrac{2}{3}}u \,\mathrm{d}x + \sqrt{\tfrac{2}{3}}\left(\eta^2 u + \tfrac{1}{3}u^3 + u_{xx} + \alpha u\right) \mathrm{d}t. \tag{21}$$

Then M^2 is a pss iff u satisfies the NLEE

$$u_t = u_{xxx} + (a + u^2)u_x, \tag{22}$$

where a is a constant.

2.6. *The IS equation*

Consider

$$\omega_1 = \left(\frac{u_x}{u} + u^2\right) \mathrm{d}x + \left(\frac{u_{xxx}}{u} + u^6 + 8u_x^2 + 5uu_{xx} + 9u^3u_x\right) \mathrm{d}t,$$
$$\omega_2 = \eta \,\mathrm{d}x + \eta\left(\frac{u_{xx}}{u} + u^4 + 4uu_x\right) \mathrm{d}t, \qquad \omega_3 = -\eta \,\mathrm{d}x - \eta\left(\frac{u_{xx}}{u} + u^4 + 4uu_x\right) \mathrm{d}t. \tag{23}$$

Then M^2 is a pss iff u satisfies IS equation

$$u_t = u_{xxx} + 3u^2u_{xx} + 9uu_x^2 + 3u^4u_x. \tag{24}$$

2.7. The CH equation

Consider

$$\omega_1 = \left(u_{xx} - u - \beta + \frac{\beta}{\eta^2} - \eta^{-2}\right) \mathrm{d}x + \left(\frac{u_x\beta}{\eta} - \frac{\beta}{\eta^2} + \eta^{-2} + u^2 - 1 + u\beta + \frac{u_x}{\eta} - uu_{xx}\right) \mathrm{d}t,$$

$$\omega_2 = \eta\, \mathrm{d}x + \left(\frac{-\beta}{\eta} - \eta u + \eta^{-1} + u_x\right) \mathrm{d}t,$$

$$\omega_3 = (u_{xx} - u + 1)\, \mathrm{d}x + \left(\frac{u\beta}{\eta^2} - \frac{u_x\beta}{\eta} - \frac{u}{\eta^2} - \frac{\beta}{\eta^2} + \eta^{-2} + u^2 - u + \frac{u_x}{\eta} - uu_{xx}\right) \mathrm{d}t. \quad (25)$$

Then M^2 is a pss iff u satisfies the CH equation

$$2u_x u_{xx} + uu_{xxx} = u_t - u_{xxt} + 3uu_x, \quad (26)$$

in which the parameters η and β are constrained by the relation

$$\eta^4 - \eta^2 + \beta^2\eta^2 = \beta^2 + 1 - 2\beta. \quad (27)$$

2.8. The HS equation

Consider

$$\omega_1 = (u_{xx} - \beta)\, \mathrm{d}x + \left(\frac{u_x - u_x\beta}{\eta} + \frac{1-\beta}{\eta^2} - uu_{xx} - 1 + u\beta\right) \mathrm{d}t,$$

$$\omega_2 = \eta\, \mathrm{d}x + \left(\frac{1-\beta}{\eta} - \eta u + u_x\right) \mathrm{d}t,$$

$$\omega_3 = (u_{xx} + 1)\, \mathrm{d}x + \left(\frac{u_x - u_x\beta}{\eta} + \frac{1-\beta}{\eta^2} - uu_{xx} - u\right) \mathrm{d}t. \quad (28)$$

Then M^2 is a pss iff u satisfies the HS equation

$$2u_x u_{xx} + u_{xxt} + uu_{xxx} = 0, \quad (29)$$

in which the parameters η and β are constrained by the relation

$$\eta^2 + \beta^2 = 1. \quad (30)$$

3. Infinite number of conservation laws for some NLEEs

In this section we apply the Cavalcante and Tenenblat method to obtain an infinite number of conserved densities for some NLEEs. To each equation we associate functions f_{ij} satisfying the following theorem [7].

Theorem 1. *Let f_{ij}, $1 \le i \le 3$, $1 \le j \le 2$, be differentiable functions of x, t such that*

$$-f_{11,t} + f_{12,x} = f_{31}f_{22} - f_{21}f_{32}, \qquad -f_{21,t} + f_{22,x} = f_{11}f_{32} - f_{12}f_{31},$$
$$-f_{31,t} + f_{32,x} = f_{11}f_{22} - f_{12}f_{21}. \tag{31}$$

Then the following statements are valid:

(i) *The following system is completely integrable for ϕ*:

$$\phi_x = f_{31} + f_{11}\sin\phi + f_{21}\cos\phi, \qquad \phi_t = f_{32} + f_{12}\sin\phi + f_{22}\cos\phi. \tag{32}$$

(ii) *For any solution ϕ of* (32)

$$\omega = (f_{11}\cos\phi - f_{21}\sin\phi)\,\mathrm{d}x + (f_{12}\cos\phi - f_{22}\sin\phi)\,\mathrm{d}t, \tag{33}$$

is a closed one-form.

(iii) *If f_{ij} are analytic functions of a parameter η at the origin, then the solutions $\phi(x, t, \eta)$ of* (32) *and the one-form ω are also analytic in η at the origin.*

Cavalcante and Tenenblat supposed [7]

$$f_{ij}(x, t, \eta) = \sum_{k=0}^{\infty} f_{ij}^{k}(x, t)\eta^{k}. \tag{34}$$

Then the solution ϕ of (32) is of the form

$$\phi(x, t, \eta) = \sum_{j=0}^{\infty} \phi_j(x, t)\eta^{j}. \tag{35}$$

We consider the following functions of η, for fixed x, t:

$$A(\eta) = \cos\phi = \cos\left(\sum_{j=0}^{\infty}\phi_j(x, t)\eta^{j}\right), \qquad B(\eta) = \sin\phi = \sin\left(\sum_{j=0}^{\infty}\phi_j(x, t)\eta^{j}\right). \tag{36}$$

It follows from (36) that

$$A(0) = \cos\phi_0, \quad B(0) = \sin\phi_0, \qquad \frac{\mathrm{d}^k A}{\mathrm{d}\eta^k}(0) = -(k-1)!\sum_{i=0}^{k-1}\frac{k-i}{i!}\frac{\mathrm{d}^i B}{\mathrm{d}\eta^i}(0)\phi_{k-i},$$
$$\frac{\mathrm{d}^k B}{\mathrm{d}\eta^k}(0) = (k-1)!\sum_{i=0}^{k-1}\frac{k-i}{i!}\frac{\mathrm{d}^i A}{\mathrm{d}\eta^i}(0)\phi_{k-i} \tag{37}$$

for $k \ge 1$. Finally, we define the functions of x, t:

$$H_k^{ij} = f_{1k}^{i}\frac{\mathrm{d}^{j-i}A}{\mathrm{d}\eta^{j-i}}(0) - f_{2k}^{i}\frac{\mathrm{d}^{j-i}B}{\mathrm{d}\eta^{j-i}}(0), \qquad L_k^{ij} = f_{1k}^{i}\frac{\mathrm{d}^{j-i}B}{\mathrm{d}\eta^{j-i}}(0) + f_{2k}^{i}\frac{\mathrm{d}^{j-i}A}{\mathrm{d}\eta^{j-i}}(0),$$
$$F_{1k} = f_{3k}^{1} + L_k^{11}, \qquad F_{nk} = f_{3k}^{n} + \sum_{r=1}^{n-1}\frac{n-r}{r!}H_k^{0r}\phi_{n-r} + \sum_{r=1}^{n}\frac{1}{n-r!}L_k^{rn}, \tag{38}$$

where i, j, n are the nonnegative integers such that $j \geq i$, $n \geq 2$, and $k = 1, 2$. The functions H_k^{ij} and L_k^{ij} defined above depend on $\phi_0, \phi_1, \dots, \phi_{j-i}$; the functions F_{1k} and F_{nk} depend on ϕ_0 and $\phi_0, \phi_1, \dots, \phi_{n-1}$, respectively.

Corollary 1. *Let $f_{ij}(x, t, \eta)$, $1 \leq i \leq 3$, $1 \leq j \leq 2$, be differentiable functions of x, t, analytic at $\eta = 0$, that satisfy (31). Then, with the above notation, the following statements hold*:

(a) *The solution ϕ of (32) is analytic at $\eta = 0$, then ϕ_0 is determined by*

$$\phi_{0,x} = f_{31}^0 + L_1^{00}, \quad \phi_{0,t} = f_{32}^0 + L_2^{00}, \tag{39}$$

and, for $j \geq 1$, ϕ_j are recursively determined by the system

$$\phi_{j,x} = H_1^{00}\phi_j + F_{j1}, \quad \phi_{j,t} = H_2^{00}\phi_j + F_{j2}. \tag{40}$$

(b) *For any such solution ϕ and integer $j \geq 1$*

$$\omega^j = \sum_{i=0}^{j} \frac{1}{(j-i)!}(H_1^{ij}\,\mathrm{d}x + H_2^{ij}\,\mathrm{d}t), \tag{41}$$

is a closed one-form.

Now we consider NLPDEs for $u(x, t)$ which describes a pss. There exist functions f_{ij}, $1 \leq i \leq 3$, $1 \leq j \leq 2$, which depend on $u(x, t)$ and its derivatives such that, for any solution u of the EE, f_{ij} satisfy (31). Then it follows from Theorem 1 that (32) is completely integrable for ϕ. Suppose f_{ij} to be analytic functions of a parameter η, then the solutions ϕ of (32) and the one-form ω, given in (33), are analytic in η. Their coefficients ϕ_j and ω^j, as functions of u, are determined in (39)–(41). Therefore the closed one-forms ω^j provide a sequence of conservation laws for the PDE, with conserved density and flux given respectively by

$$D_j = \sum_{i=0}^{j} \frac{1}{(j-i)!} H_1^{ij}, \qquad F_j = -\sum_{i=0}^{j} \frac{1}{(j-i)!} H_2^{ij}, \quad j \geq 0. \tag{42}$$

We consider the following examples.

3.1. Nonlinear Schrödinger equation

For Eq. (14) we consider the following functions of $u(x, t)$ defined by:

$$f_{11} = \tfrac{1}{2}(u - u^*), \quad f_{12} = [i\eta(u - u^*) + i(u_x + u_x^*)], \quad f_{21} = \eta,$$
$$f_{22} = [2i\eta^2 + iuu^*], \quad f_{31} = \tfrac{-1}{2}(u + u^*), \quad f_{32} = [-i\eta(u + u^*) + i(u_x^* - u_x)], \tag{43}$$

as corresponding to Eq. (13). For any solution u of Eq. (14) the above functions f_{ij} satisfy (31). Applying the corollary, we have a sequence of functions ϕ_j determined in (36) and

(37). It follows from (43) that (39) reduces to

$$\phi_{0,x} = \frac{-1}{2}(u + u^*) + \frac{1}{2}(u - u^*)\sin\phi_0,$$
$$\phi_{0,t} = i(u_x^* - u_x) + i(u_x^* + u_x)\sin\phi_0 + 2iuu^*\cos\phi_0, \tag{44}$$

and from (37) we obtain recursively

$$\phi_j = e^s\left(1 + \int F_{j1}e^{-s}\,\mathrm{d}x\right), \quad j \geq 1, \tag{45}$$

where

$$s = \int \frac{1}{2}(u - u^*)\cos\phi_0\,\mathrm{d}x,$$

and

$$F_{j1} = \frac{1}{(j-1)!}\frac{\mathrm{d}^{j-1}A}{\mathrm{d}\eta^{j-1}}(0) + \frac{1}{j}\sum_{i=1}^{j-1}\frac{j-i}{i!}\frac{(u-u^*)}{2}\frac{\mathrm{d}^iA}{\mathrm{d}\eta^i}(0)\phi_{j-i}.$$

The sequence of conserved densities for NLSE is given by

$$\frac{1}{2}(u - u^*)\cos\phi_0, \qquad \frac{1}{j!}\frac{(u-u^*)}{2}\frac{\mathrm{d}^jA}{\mathrm{d}\eta^j}(0) - \frac{1}{(j-1)!}\frac{\mathrm{d}^{j-1}B}{\mathrm{d}\eta^{j-1}}(0), \quad j \geq 1. \tag{46}$$

Solving the integrable system of Eq. (44), then from ϕ_0 we obtain ϕ_j, $j \geq 1$, defined in (45).

3.2. Liouville's equation

For Eq. (16) we consider the functions defined by

$$f_{11} = u_x, \quad f_{12} = 0, \quad f_{21} = \eta, \quad f_{22} = \frac{e^u}{\eta}, \quad f_{31} = 0, \quad f_{32} = \frac{e^u}{\eta}. \tag{47}$$

For any solution u of Eq. (16), the above functions satisfy (31). As in the preceding example, we obtain a sequence of conserved densities for (16) by using theorem (1). Substituting (47) into (32), we obtain the system of equations

$$\phi_x = u_x\sin\phi + \eta\cos\phi, \qquad \phi_t = \frac{e^u}{\eta}(1 + \cos\phi), \tag{48}$$

which is completely integrable whenever u is a solution of Eq. (16). From the first equation of (48) we conclude that ϕ is analytic with respect to η. Therefore, consider

$$\phi = \sum_{j=0}^{\infty}\phi_j(x,t)\eta^j. \tag{49}$$

Eq. (48) reduces to

$$\phi_{0,x} = u_x\sin\phi_0, \quad \phi_0 = n\pi, \quad n = 0, 3, 5, \ldots,$$
$$\phi_j = e^h\left(1 + \int F_{j1}e^{-h}\,\mathrm{d}x\right), \quad j \geq 1, \tag{50}$$

where

$$h = -\int u_x \,\mathrm{d}x = -u + g(t), \qquad F_{11} = -1,$$

and

$$F_{j1} = \frac{1}{(j-1)!}\frac{\mathrm{d}^{j-1}A}{\mathrm{d}\eta^{j-1}}(0) + \frac{1}{j}\sum_{i=1}^{j-1}\frac{j-i}{i!}u_x\frac{\mathrm{d}^i A}{\mathrm{d}\eta^i}(0)\phi_{j-i}, \qquad j \geq 2.$$

Using (37) in the above expression, we obtain ϕ_j in terms of u. We display only the first terms of the series:

$$\phi_0 = n\pi, \quad n = 0, 3, 5, \dots, \qquad \phi_1 = e^h\left(1 + \int e^{-h}\,\mathrm{d}x\right), \qquad \phi_2 = e^h, \quad \text{etc.} \tag{51}$$

The sequence of conserved densities for Liouville's equation is given by

$$u_x \cos\phi_0, \qquad \frac{u_x}{j!}\frac{\mathrm{d}^j A}{\mathrm{d}\eta^j}(0) - \frac{1}{(j-1)!}\frac{\mathrm{d}^{j-1}B}{\mathrm{d}\eta^{j-1}}(0), \qquad j \geq 1. \tag{52}$$

Using the expressions in Eq. (37) and the functions ϕ_j given in (50) we obtain the first conserved densities:

$$-u_x, u_x e^{-2u}(2u_t^2 + 2u_{tt} + e^u u_t), \quad \text{etc.} \tag{53}$$

3.3. The family of equations

For any solution u of the family of Eq. (18), the functions

$$f_{11} = 0, \quad f_{12} = -\frac{\xi}{\eta}g', \quad f_{21} = \eta, \quad f_{22} = \frac{\xi^2 g - \theta}{\eta} + \beta\eta,$$
$$f_{31} = \xi u_x, \quad f_{32} = \xi(\alpha g + \beta)u_x, \tag{54}$$

satisfy (31). Applying the corollary, we obtain ϕ_j, $j \geq 0$, defined by

$$\phi_0 = \int \xi u_x \,\mathrm{d}x = \xi u + h(t), \qquad \phi_j = \frac{1}{(j-1)!}\int \frac{\mathrm{d}^{j-1}A}{\mathrm{d}\eta^{j-1}}(0)\,\mathrm{d}x, \qquad j \geq 1. \tag{55}$$

Using (37) in the above expressions we obtain ϕ_j. The first terms are

$$\phi_0 = \xi u + h(t), \qquad \phi_1 = \int \cos\phi_0 \,\mathrm{d}x, \qquad \phi_2 = -\int \phi_1 \sin\phi_0 \,\mathrm{d}x, \quad \text{etc.} \tag{56}$$

The conserved densities are given by

$$\frac{\mathrm{d}^{j-1}B}{\mathrm{d}\eta^{j-1}}(0), \qquad j \geq 1. \tag{57}$$

Using (37), we obtain the first terms

$$\sin\phi_0, \phi_1\cos\phi_0, 2\phi_2\cos\phi_0 - \phi_1^2\sin\phi_0, \quad \text{etc.}, \tag{58}$$

where the ϕ_j are given in (55).

3.4. The family of equations

For Eq. (20) we consider the functions of $u(x,t)$ defined by

$$f_{11}=\xi u_x,\quad f_{12}=\xi(\alpha g+\beta)u_x,\quad f_{21}=\eta,$$
$$f_{22}=\frac{\xi^2 g-\theta}{\eta}+\beta\eta,\quad f_{31}=0,\quad f_{32}=\frac{\xi}{\eta}g'. \tag{59}$$

For any solution u of Eq. (20), the above functions f_{ij} satisfy (31). Applying the corollary, we have a sequence of functions ϕ_j determined in (36) and (38). It follows from (59) that reduces to

$$\phi_{0,x}=\xi u_x\sin\phi_0,\qquad \xi g'+(\xi^2 g-\theta)\cos\phi_0=0, \tag{60}$$

and from (37) we obtain recursively

$$\phi_j=e^h\left(1+\int F_{j1}e^{-h}\,\mathrm{d}x\right),\quad j\geq 1, \tag{61}$$

where

$$h=\int \xi u_x\cos\phi_0\,\mathrm{d}x,$$

and

$$F_{j1}=\frac{1}{(j-1)!}\frac{\mathrm{d}^{j-1}A}{\mathrm{d}\eta^{j-1}}(0)+\frac{1}{j}\sum_{i=1}^{j-1}\frac{j-i}{i!}\xi u_x\frac{\mathrm{d}^iA}{\mathrm{d}\eta^i}(0)\phi_{j-i}.$$

The sequence of conserved densities for the family of equations is given by

$$\xi u_x\cos\phi_0,\ \frac{\xi u_x}{j!}\frac{\mathrm{d}^jA}{\mathrm{d}\eta^j}(0)-\frac{1}{(j-1)!}\frac{\mathrm{d}^{j-1}B}{\mathrm{d}\eta^{j-1}}(0),\quad j\geq 1. \tag{62}$$

Solving the integrable system of Eq. (60), from ϕ_0 we obtain ϕ_j, $j\geq 1$, defined in (61).

3.5. NLEE

For any solution u of the NLEE (22) the functions

$$f_{11}=0,\quad f_{12}=-\eta\sqrt{\tfrac{2}{3}}u_x,\quad f_{21}=\eta,\quad f_{22}=(\eta^3+\tfrac{1}{3}\eta u^2+a\eta),$$
$$f_{31}=\sqrt{\tfrac{2}{3}}u,\quad f_{32}=\sqrt{\tfrac{2}{3}}(\eta^2u+\tfrac{1}{3}u^3+u_{xx}+au), \tag{63}$$

satisfy (31). Applying the corollary, we obtain ϕ_j, $j\geq 0$, defined by

$$\phi_0=\sqrt{\frac{2}{3}}\int u\,\mathrm{d}x,\qquad \phi_j=\frac{1}{(j-1)!}\int\frac{\mathrm{d}^{j-1}A}{\mathrm{d}\eta^{j-1}}(0)\,\mathrm{d}x,\quad j\geq 1. \tag{64}$$

Using (37) in the above expressions we obtain ϕ_j. The first terms are

$$\phi_0 = \sqrt{\frac{2}{3}} \int u \,\mathrm{d}x, \qquad \phi_1 = \int \cos\phi_0 \,\mathrm{d}x, \qquad \phi_2 = -\int \phi_1 \sin\phi_0 \,\mathrm{d}x, \quad \text{etc.}$$

The conserved densities are given by

$$\frac{\mathrm{d}^{j-1}B}{\mathrm{d}\eta^{j-1}}(0), \qquad j \geq 1.$$

Using (37), we obtain the first ones

$$\sin\phi_0,\, \phi_1\cos\phi_0,\, 2\phi_2\cos\phi_0 - \phi_1^2\sin\phi_0, \quad \text{etc.}, \tag{65}$$

where the ϕ_j are given in (64).

4. Conservation laws for NLEEs which describe pss

One of the most widely accepted definitions of integrability of PDEs requires the existence of soliton solutions, i.e., of a special kind of traveling wave solutions that interact "elastically", without changing their shapes. The analytic construction of soliton solutions is based on the general ISM. In the formulation of Zakharov and Shabat [19], all known integrable systems supporting solitons can be realized as the integrability condition of a linear problem of the form

$$\nu_x = P\nu, \quad \nu_t = Q\nu, \tag{66}$$

where the matrices P and Q are two 2×2 null-trace matrices

$$P = \begin{pmatrix} \frac{\eta}{2} & q \\ r & \frac{-\eta}{2} \end{pmatrix}, \quad Q = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}. \tag{67}$$

Here η is a parameter, independent of x and t. Thus, an equation is kinematically integrable if it is equivalent to the curvature condition

$$P_x - Q_t + [P, Q] = 0. \tag{68}$$

Konno and Wadati introduced the function [18]

$$\Gamma = \frac{\nu_1}{\nu_2}, \tag{69}$$

and for each of the NLEE, derived a BT with the following form:

$$u = u_0 + f(\Gamma, \eta), \tag{70}$$

where u is a new solution of the corresponding NLEE. As mentioned in the previous sections, Sasaki [1], Chern and Tenenblat [3], and Cavalcante and Tenenblat [7] have given a geometrical method for constructing conservation laws of equations describing pss. The formal content of this method is contained in the following theorem, which may be seen as generalizing the classical discussion on conservation laws appearing by Wadati et al. [20].

Theorem 2. *Suppose that* $u_t = F(u, u_x, \dots, u_{x^k})$ *or more generally* $F(x, t, u, u_x, \dots, u_{x^n t^m}) = 0$ *is an EE describing pss. The systems*

$$D_x\phi_1 = qr + \left(\frac{D_x q}{q} - \eta\right)\phi_1 - \phi_1^2, \tag{71}$$

$$D_t\left(\frac{\eta}{2} + \phi_1\right) = D_x\left(A + \frac{B}{q} - \phi_1\right), \tag{72}$$

and

$$D_x\phi_2 = -qr + \left(\frac{D_x r}{r} + \eta\right)\phi_2 + \phi_2^2, \tag{73}$$

$$D_t\left(\frac{\eta}{2} + \phi_2\right) = D_x\left(A + \frac{C}{r}\phi_2\right), \tag{74}$$

in which D_x, *and* D_t *are the total derivative operator defined by*

$$D_x = \frac{\partial}{\partial x} + \sum_{k=0}^{\infty} u_{k+1}\frac{\partial}{\partial u_k}, \qquad D_t = \frac{\partial}{\partial t} + \sum_{k=0}^{\infty} D_x^k(f)\frac{\partial}{\partial u_k},$$

are integrable on solutions of the equation $u_t = F(u, u_x, \dots, u_{x^k})$ *or generally* $F(x, t, u, u_x, \dots, u_{x^n t^m}) = 0$.

Proof. The equation $u_t = F(u, u_x, \dots, u_{x^k})$ or more generally $F(x, t, u, u_x, \dots, u_{x^n t^m}) = 0$ is the necessary and sufficient condition for the integrability of the linear problem (66). Equivalently, in (68), the functions r, q, A, B and C satisfy the equations

$$A_x = qC - rB, \tag{75}$$

$$q_t - 2Aq - B_x + \eta\beta = 0, \tag{76}$$

$$C_x = r_t + 2Ar - \eta C. \tag{77}$$

Set

$$\eta = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix},$$

and define $\phi_1 = q/\Gamma, \phi_2 = r\Gamma$. Straightforward computations using Eqs. (75)–(77) allow one to check that if

$$\nu = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix},$$

is a nontrivial solution of the linear system $d\nu = \Omega\nu$, ϕ_1 is a solution of the system (71) and (72) and ϕ_2 is a solution of the system of Eqs. (73) and (74). □

This theorem provides one with at least one η-dependent conservation law of the EE $u_t = F(u, u_x, \dots, u_{x^k})$ or $F(x, t, u, u_x, \dots, u_{x^n t^m}) = 0$, to wit, Eqs. (71) and (72) or ((73) and (74)). One obtains a sequence of η-independent conservation laws by expanding ϕ_1 or ϕ_2 in inverse powers of η [9]

$$\phi_2 = \sum_{n=1}^{\infty} \phi_2^{(n)} \eta^{-n}, \tag{78}$$

consideration of Eq. (73) yields the recursion relation

$$\phi_2^{(1)} = -qr, \tag{79}$$

$$\phi_2^{(n+1)} = \frac{D_x r}{r} \phi_2^{(n)} + D_x \phi_2^{(n)} + \sum_{i=1}^{n-1} \phi_2^{(i)} \phi_2^{(n-i)}, \quad n \geq 1, \tag{80}$$

which in turn, by replacing into (74), yields the sequence of conservation laws of equations integrable by AKNS inverse scattering found by Wadati et al. [20]. This section ends with the examples.

4.1. Nonlinear Schrödinger equation

For Eq. (14) we consider the functions of $u(x, t)$ defined by

$$\begin{aligned} &r = \tfrac{-1}{2} u^*, \quad q = \tfrac{1}{2} u, \quad A - i\eta^2 + \tfrac{1}{2} i u u^*, \\ &B = [i\eta u + i u_x], \quad C = -i\eta(u^*) + i(u_x^*). \end{aligned} \tag{81}$$

Eq. (73) becomes

$$D_x \phi_2 = \frac{1}{4} u u^* + \left(\frac{D_x u^*}{u^*} + \eta \right) \phi_2 + \phi_2^2. \tag{82}$$

Assume that ϕ_2 can be expanded in a series of the form (78).

Eq. (81) implies that ϕ_2 is determined by the recursion relation

$$\phi_2^{(1)} = \tfrac{1}{4} u u^*, \tag{83}$$

$$\phi_2^{(n+1)} = \frac{D_x u^*}{u^*} \phi_2^{(n)} + D_x \phi_2^{(n)} + \sum_{i=1}^{n-1} \phi_2^{(i)} \phi_2^{(n-i)}, \quad n \geq 1, \tag{84}$$

whenever $u(x, t)$ is a solution of the NLSE. This recursion relation yields a sequence of conserved densities given by the coefficients of the series in η

$$\frac{\eta}{2} + \sum_{n=1}^{\infty} \phi_2^{(n)} \eta^{-n}, \tag{85}$$

which one obtains from Eq. (74).

4.2. Liouville's equation

For (16) we consider the functions defined by

$$r = \frac{1}{2}u_x, \quad q = \frac{1}{2}u_x, \quad A = \frac{e^u}{2\eta}, \quad B = \frac{-e^u}{2\eta}, \quad C = \frac{e^u}{2\eta}. \tag{86}$$

Eq. (76) becomes

$$D_x\phi_2 = \frac{-1}{4}u_x^2 + \left(\frac{D_x u_x}{u_x} + \eta\right)\phi_2 + \phi_2^2. \tag{87}$$

Assume that ϕ_2 can be expanded in a series of the form (78). Eq. (87) implies that ϕ_2 is determined by the recursion relation

$$\phi_2^{(1)} = \tfrac{-1}{4}u_x^2, \tag{88}$$

$$\phi_2^{(n+1)} = \frac{D_x u_x}{u_x}\phi_2^{(n)} + D_x\phi_2^{(n)} + \sum_{i-1}^{n-1}\phi_2^{(i)}\phi_2^{(n-i)}, \quad n \geq 1, \tag{89}$$

whenever $u(x, t)$ is a solution of Liouville's equation, and it follows from Eq. (74) that the coefficients of the series in η, given in (85) are a sequence of conserved densities for Liouville's equation.

4.3. The family of equations

For any solution u of the family of Eq. (18), we consider the functions

$$r = \frac{\xi u_x}{2}, \quad q = \frac{-\xi u_x}{2}, \quad A = \frac{1}{2}\left(\frac{\xi^2 g - \theta}{\eta} + \beta\eta\right),$$
$$B = -\frac{1}{2}\left(\frac{\xi}{\eta}g' + \xi(\alpha g + \beta)u_x\right), \quad C = \frac{1}{2}\left(\xi(\alpha g + \beta)u_x - \frac{\xi}{\eta}g'\right). \tag{90}$$

Eq. (73) becomes

$$D_x\phi_2 = \frac{1}{4}\xi^2 u_x^2 + \left(\frac{D_x u_x}{u_x} + \eta\right)\phi_2 + \phi_2^2. \tag{91}$$

Assume that ϕ_2 can be expanded in a series of the form (78). Eq. (91) implies that ϕ_2 is determined by the recursion relation

$$\phi_2^{(1)} = \tfrac{1}{4}\xi^2 u_x^2, \tag{92}$$

$$\phi_2^{(n+1)} = \frac{D_x u_x}{u_x}\phi_2^{(n)} + D_x\phi_2^{(n)} + \sum_{i=1}^{n-1}\phi_2^{(i)}\phi_2^{(n-i)}, \quad n \geq 1, \tag{93}$$

whenever $u(x, t)$ is a solution of the family of equations, and it follows from Eq. (74) that the coefficients of the series in η, given in (85) are a sequence of conserved densities for the family of equations.

4.4. The family of equations

For the Eq. (20) we consider the functions of $u(x,t)$ defined by

$$r=\frac{\xi u_x}{2},\quad q=\frac{\xi u_x}{2},\quad A=\frac{1}{2}\left(\frac{\xi^2 g-\theta}{\eta}+\beta\eta\right),\quad B=\frac{1}{2}\left(\frac{\xi}{\eta}g'+\xi(\alpha g+\beta)u_x\right),$$
$$C=\frac{1}{2}\left(\xi(\alpha g+\beta)u_x-\frac{\xi}{\eta}g'\right). \tag{94}$$

Eq. (73) becomes

$$D_x\phi_2=\frac{-1}{4}\xi^2u_x^2+\left(\frac{D_xu_x}{u_x}+\eta\right)\phi_2+\phi_2^2. \tag{95}$$

Assume that ϕ_2 can be expanded in a series of the form (78). Eq. (95) implies that ϕ_2 is determined by the recursion relations

$$\phi_2^{(1)}=\frac{-1}{4}\xi^2u_x^2,\qquad \phi_2^{(n+1)}=\frac{D_xu_x}{u_x}\phi_2^{(n)}+D_x\phi_2^{(n)}+\sum_{i=1}^{n-1}\phi_2^{(i)}\phi_2^{(n-i)},\quad n\geq 1, \tag{96}$$

whenever $u(x,t)$ is a solution of the family of equations, and it follows from Eq. (74) that the coefficients of the series in η, given in (85), are a sequence of conserved densities for the family of equations.

4.5. The NLEE

For any solution u of the NLEE (22) the functions

$$r=\frac{u}{\sqrt{6}},\quad q=\frac{-u}{\sqrt{6}},\quad A=\frac{1}{2}\left(\eta^3+\frac{\eta u^2}{3}+a\eta\right),$$
$$B=\frac{1}{\sqrt{6}}\left(-\eta u_x-\eta^2u-\frac{u^3}{3}-u_{xx}-au\right),$$
$$C=\frac{1}{\sqrt{6}}\left(-\eta u_x+\eta^2u+\frac{u^3}{3}+u_{xx}+au\right). \tag{97}$$

Eq. (73) becomes

$$D_x\phi_2=\frac{1}{6}u^2+\left(\frac{D_xu}{u}+\eta\right)\phi_2+\phi_2^2. \tag{98}$$

Assume that ϕ_2 can be expanded in a series of the form (78). Eq. (98) implies that ϕ_2 is determined by the recursion relation

$$\phi_2^{(1)}=\tfrac{1}{6}u^2, \tag{99}$$

$$\phi_2^{(n+1)}=\frac{D_xu}{u}\phi_2^{(n)}+D_x\phi_2^{(n)}+\sum_{i=1}^{n-1}\phi_2^{(i)}\phi_2^{(n-i)},\quad n\geq 1, \tag{100}$$

whenever $u(x,t)$ is a solution of the NLEE, and it follows from Eq. (74) that the coefficients of the series in η, given in (85) are a sequence of conserved densities for the NLEE.

4.6. The IS equation

For any solution u of the IS Eq. (24) we consider the functions

$$r=\frac{1}{2}\left(\frac{u_x}{u}+u^2-\eta\right),\quad q=\frac{1}{2}\left(\frac{u_x}{u}+u^2+\eta\right),\quad A=\frac{\eta}{2}\left(\frac{u_{xx}}{u}+u^4+4uu_x\right),$$
$$B=\frac{1}{2}\left[\left(\frac{u_{xxx}}{u}+u^6+8u_x^2+5uu_{xx}+9u^3u_x\right)+\eta\left(\frac{u_{xx}}{u}+u^4+4uu_x\right)\right],$$
$$C=\frac{1}{2}\left[\left(\frac{u_{xxx}}{u}+u^6+8u_x^2+5uu_{xx}+9u^3u_x\right)-\eta\left(\frac{u_{xx}}{u}+u^4+4uu_x\right)\right]. \tag{101}$$

Eq. (73) becomes

$$D_x\phi_2=\frac{-1}{4}\left(\left(\frac{u_x}{u}+u^2\right)^2-\eta^2\right)+\left(\frac{D_x(u_x/u+u^2-\eta)}{u_x/u+u^2-\eta}+\eta\right)\phi_2+\phi_2^2. \tag{102}$$

Assume that ϕ_2 can be expanded in a series of the form (78). Eq. (102) implies that ϕ_2 is determined by the recursion relation

$$\phi_2^{(1)}=\frac{-1}{4}\left(\left(\frac{u_x}{u}+u^2\right)^2-\eta^2\right), \tag{103}$$

$$\phi_2^{(n+1)}=\frac{D_x(u_x/u+u^2-\eta)}{u_x/u+u^2-\eta}\phi_2^{(n)}+D_x\phi_2^{(n)}+\sum_{i=1}^{n-1}\phi_2^{(i)}\phi_2^{(n-i)},\quad n\geq 1, \tag{104}$$

whenever $u(x,t)$ is a solution of the IS equation, and it follows from Eq. (74) that the coefficients of the series in η, given in (85), are a sequence of conserved densities for the IS equation.

4.7. CH equation

For any solution u of the CH Eq. (26), we consider the functions

$$r=\left(u_{xx}-u-\frac{\beta}{2}+\frac{\beta}{2\eta^2}-\frac{\eta^{-2}}{2}+\frac{1}{2}\right),\quad q=\frac{1}{2}\left(\frac{\beta}{\eta^2}-\eta^{-2}-\beta-1\right),$$
$$A=\frac{1}{2}\left(\frac{-\beta}{\eta}-\eta u+\eta^{-1}+u_x\right),\quad B=\left(\frac{u_x\beta}{\eta}+\frac{u+u\beta-1}{2}-\frac{u\beta}{2\eta^2}\right),$$
$$C=\left(u^2+\eta^{-2}-uu_{xx}-\frac{\beta}{\eta^2}-\frac{u_x}{\eta}-\frac{u}{2}(\beta-1)\left(1+\frac{1}{\eta^2}\right)-\frac{1}{2}\right), \tag{105}$$

in which the parameters η and β are constrained by the relation (27). Eq. (73) becomes

$$D_x\phi_2=\frac{-1}{2}\left(u_{xx}-u-\frac{\beta}{2}+\frac{\beta}{2\eta^2}-\frac{\eta^{-2}}{2}+\frac{1}{2}\right)\left(\frac{\beta}{\eta^2}-\eta^{-2}-\beta-1\right)$$
$$+\left(\frac{D_x(u_{xx}-u)}{u_{xx}-u-\beta/2+\beta/2\eta^2-\eta^{-2}/2+1/2}+\eta\right)\phi_2+\phi_2^2. \tag{106}$$

Assume that ϕ_2 can be expanded in a series of the form (78). Eq. (106) implies that ϕ_2 is determined by the recursion relation

$$\phi_2^{(1)} = \frac{-1}{2}\left(u_{xx} - u - \frac{\beta}{2} + \frac{\beta}{2\eta^2} - \frac{\eta^{-2}}{2} + \frac{1}{2}\right)\left(\frac{\beta}{\eta^2} - \eta^{-2} - \beta - 1\right), \tag{107}$$

$$\phi_2^{(n+1)} = \frac{D_x(u_{xx} - u)}{u_{xx} - u - \beta/2 + \beta/2\eta^2 - \eta^{-2}/2 + 1/2}\phi_2^{(n)} + D_x\phi_2^{(n)} + \sum_{i=1}^{n-1}\phi_2^{(i)}\phi_2^{(n-i)}, \quad n \geq 1, \tag{108}$$

whenever $u(x, t)$ is a solution of the CH equation, and it follows from Eq. (74) that the coefficients of the series in η, given in (85), are a sequence of conserved densities for the CH equation.

4.8. HS equation

For any solution u of the HS Eq. (29), we consider the functions

$$r = \left(u_{xx} - \frac{\beta}{2} + \frac{1}{2}\right), \quad q = \frac{-1}{2}(\beta + 1), \quad A = \frac{1}{2}\left(\frac{1-\beta}{\eta} - \eta u + u_x\right),$$
$$B = \frac{1}{2}(u\beta - 1 + u), \quad C = \left(\frac{u_x - u_x\beta}{\eta} + \frac{1-\beta}{\eta^2} - uu_{xx} + \frac{u\beta - u - 1}{2}\right), \tag{109}$$

in which the parameters η and β are constrained by the relation (30). Eq. (73) becomes

$$D_x\phi_2 = \frac{1}{2}\left(u_{xx} - \frac{\beta}{2} + \frac{1}{2}\right)(\beta + 1) + \left(\frac{D_x u_{xx}}{u_{xx} - \beta/2 + 1/2} + \eta\right)\phi_2 + \phi_2^2. \tag{110}$$

Assume that ϕ_2 can be expanded in a power series of the form (78). Eq. (110) implies that ϕ_2 is determined by the recursion relations

$$\phi_2^{(1)} = \frac{1}{2}\left(u_{xx} - \frac{\beta}{2} + \frac{1}{2}\right)(\beta + 1), \tag{111}$$

$$\phi_2^{(n+1)} = \frac{D_x u_{xx}}{u_{xx} - \beta/2 + 1/2}\phi_2^{(n)} + D_x\phi_2^{(n)} + \sum_{i=1}^{n-1}\phi_2^{(i)}\phi_2^{(n-i)}, \quad n \geq 1, \tag{112}$$

whenever $u(x, t)$ is a solution of the HS equation, and it follows from Eq. (74) that the coefficients of the series in η, given in (85), are a sequence of conserved densities for the HS equation.

5. Conclusion

The inverse scattering method [18–20] may be rewritten by considering ν as a three component vector and Ω as a traceless 3×3 matrix one-form [17]. The latter yields directly the curvature condition (Gaussian curvature equal to -1, corresponding to pseudo-spherical surfaces). This geometrical method is considered for several NLPDEs which describe pss: NLSE, Liouville's equation, the two families of equations, a NLEE, the IS, CH and HS equations. Next an infinite number of conservation laws is derived for the first five of the NLPDEs just mentioned using a theorem by Cavalcante and Tenenblat [7]. This geometrical method allows some further generalization of the work on conservation laws given by Wadati et al. [20]. An infinite number of conservation laws for all eight NLPDEs mentioned above are derived in this way.

Acknowledgements

It is a pleasure to thank Dr. G.K. Karugila (University of Antwerp, Belgium) for his kind assistance. One of us (AHK) is indebted to the University of Antwerp for kind hospitality. We like to thank the referee for his comments.

References

[1] R. Sasaki, Soliton equations and pseudo-spherical surfaces, Nucl. Phys. B 154 (1979) 343–357.
[2] M.J. Ablowitz, D.J. Kaup, A.C. Newell, H. Segur, The inverse scattering transform-Fourier analysis for nonlinear problems, Stud. Appl. Math. 53 (1974) 243.
[3] S.S. Chern, K. Tenenblat, Pseudo-spherical surfaces and evolution equations, Appl. Math. 74 (1986) 1.
[4] L.P. Jorge, K. Tenenblat, Linear problems associated to evolution equations of type $u_{tt} = F(u, u_x, u_{xx}, u_t)$, Stud. Appl. Math. 77 (1987) 103.
[5] M.L. Rablo, On evolution equations which describe pseudo-spherical surfaces, Stud. Appl. Math. 81 (1989) 221.
[6] M.L. Rablo, K. Tenenblat, On equations of type $u_{xt} = F(u, u_x)$ which describe pseudo-spherical surfaces, J. Math. Phys. 31 (1990) 1400.
[7] J.A. Cavalcante, K. Tenenblat, Conservation laws for nonlinear evolution equations, J. Math. Phys. 29 (1988) 1044–1049.
[8] E.G. Reyes, Pseudo-spherical surfaces and integrability of evolution equations, J. Diff. Equ. 147 (1998) 195–230.
[9] E.G. Reyes, Conservation laws and Calapso–Guichard deformations of equations describing pseudo-spherical surfaces, J. Math. Phys. 41 (2000) 2968.
[10] R. Beals, M. Rabelo, K. Tenenblat, Bäcklund transformations and inverse scattering solutions for some pseudo-spherical surfaces, Stud. Appl. Math. 81 (1989) 125.
[11] N. Kamran, K. Tenenblat, On differential equations describing pseudo-spherical surfaces, J. Diff. Equ. 104 (1995) 60–116.
[12] M. Crampin, Solitons and $SL(2, R)$, Phys. Lett. A 66 (1978) 170–172.
[13] Q. Ding, The NLS-equation and its $SL(2, R)$ structure, J. Phys. A: Math. Gen. 33 (2000) 325–329.
[14] Q. Ding, K. Tenenblat, On differential systems describing surfaces of constant curvature, J. Diff. Equ. 184 (2002) 185.
[15] Q. Ding, K. Tenenblat, Gauge equivalence of differential equations describing surfaces of constant Gaussian curvature, Int. J. Diff. Equ. Appl. 4 (2002) 273.
[16] M. Marvan, Scalar second-order evolution equations possessing an irreducible SL2-valued zero-curvature representation, J. Phys. A 36 (2002) 9431.
[17] A.S. Shchepetilov, The geometric sense of R. Sasaki connection, J. Phys. A: Math. Gen. 36 (2003) 3893.
[18] K. Konno, M. Wadati, Simple derivation of Bäcklund transformation from Riccati form of inverse method, Prog. Theor. Phys. 53 (1975) 1652.

[19] V.E. Zakharov, A.B. Shabat, A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem: I, Func. Anal. Appl. 8 (1974) 226.
[20] M. Wadati, H. Sanuki, K. Konno, Relationships among inverse method, Bäcklund transformation and an infinite number of conservation laws, Prog. Theor. Phys. 53 (1975) 419.
[21] E.G. Reyes, Geometric integrability of the Camassa–Holm equation, Lett. Math. Phys. 59 (2002) 117.
[22] E.G. Reyes, Transformations of solutions for equations and hierarchies of pseudo-spherical type, J. Phys. A: Math. Gen. 36 (2003) 125.
[23] A.H. Khater, O.H. El-Kalaawy, D.K. Callebaut, Bäcklund transformations for Alfvén solitons in a relativistic electron–positron plasma, Phys. Scripta 58 (1998) 545–548.
[24] A.H. Khater, D.K. Callebaut, O.H. El-Kalaawy, Bäcklund transformations and exact soliton solutions for nonlinear Schrödinger-type equations, Nuovo Cimento B 113 (1998) 1121–1136.
[25] A.H. Khater, D.K. Callebaut, R.S. Ibrahim, Bäcklund transformations and Painlevé analysis: exact solutions for the unstable nonlinear Schrödinger equation modelling electron-beam plasma, Phys. Plasmas 5 (1998) 395–400.
[26] A.H. Khater, D.K. Callebaut, A.R. Seadawy, Nonlinear dispersive instabilities in Kelvin–Helmholtz magnetohydrodynamic flows, Phys. Scripta 67 (2003) 340–349.
[27] A.H. Khater, D.K. Callebaut, A.R. Seadawy, General soliton solutions of an n-dimensional complex Ginzberg–Landau equation, Phys. Scripta 62 (2000) 353–357.
[28] A.H. Khater, D.K. Callebaut, W. Malfliet, A.R. Seadawy, Nonlinear dispersive instabilities in Rayleigh–Taylor magnetohydrodynamic flows, Phys. Scripta 64 (2001) 533–547.
[29] A.H. Khater, D.K. Callebaut, E.S. Kamel, Painlevé analysis and non-linear periodic solutions for isothermal magnetostatic atmospheres, Solar Phys. 178 (1998) 285–315.
[30] A.H. Khater, W. Malfliet, D.K. Callebaut, E.S. Kamel, Travelling wave solutions of some classes of nonlinear evolution equations in $(1 + 1)$ and $(2 + 1)$ dimensions, J. Comput. Appl. Math. 140 (2002) 469–477.